Key

Math 4
Final Exam Review

Name______________________________

Date_________

Prove each.

1. $\sin x \tan x + \cos x = \sec x$

$$= \frac{\sin x \cdot \sin x}{\cos x} + \cos x \cdot \frac{\cos x}{\cos x}$$

$$= \frac{\sin^2 x}{\cos x} + \frac{\cos^2 x}{\cos x}$$

$$= \frac{\sin^2 x + \cos^2 x}{\cos x}$$

$$= \frac{1}{\cos x}$$

$$= \boxed{\sec x}$$

2. $\dfrac{\sin x + \cot x}{\cos x} = \tan x + \csc x$

$$= \frac{\sin x}{\cos x} + \frac{\frac{\cos x}{\sin x}}{\cos x}$$

$$= \tan x + \frac{\cancel{\cos x}}{\sin x} \cdot \frac{1}{\cancel{\cos x}}$$

$$= \boxed{\tan x + \csc x}$$

Solve.

3. $2\sin^2 x - \sin x - 1 = 0$

$$(2\sin x + 1)(\sin x - 1) = 0$$

$2\sin x + 1 = 0$ $\quad$ $\sin x - 1 = 0$

$\sin x = \frac{-1}{2}$ $\quad$ $\sin x = 1$

$\boxed{x = \frac{7\pi}{6}, \frac{11\pi}{6}}$ $\quad$ $\boxed{x = \frac{\pi}{2}}$

4. $1 - 2\sin^2 x + 5\cos x = 2$

$$1 - 2(1 - \cos^2 x) + 5\cos x = 2$$

$$-1 - 2 + 2\cos^2 x + 5\cos x = 0$$

$$2\cos^2 x + 5\cos x - 3 = 0$$

$$(2\cos x - 1)(\cos x + 3) = 0$$

$\cos x = \frac{1}{2}$ $\quad$ $\cancel{\cos x = -3}$

$\boxed{x = \frac{\pi}{3}, \frac{5\pi}{3}}$

Simplify.

5. $\cos\left(\frac{18\pi}{12}\right)$

$$= \cos \frac{3\pi}{2}$$

$$= \boxed{0}$$

6. $\dfrac{\tan(85^\circ) - \tan(40^\circ)}{1 + \tan(85^\circ) \cdot \tan(40^\circ)}$

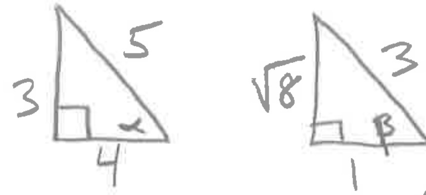
$$= \tan(85 - 40)$$

$$= \tan(45^\circ)$$

$$= \boxed{1}$$

Let $\sin\alpha = -\frac{3}{5}$ with $\frac{3\pi}{2} < \alpha < 2\pi$ and $\cos\beta = \frac{1}{3}$ with β in quadrant I.

4th quadrant



$\cos\alpha = \frac{4}{5} \rightarrow$ cosine is positive in 4th quadrant

$\sin\beta = \frac{\sqrt{8}}{3}$

$1^2 + b^2 = 3^2$

$b^2 = 8$

$b = \sqrt{8}$

Evaluate ………

7. $\cos(\alpha + \beta)$

$= \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta$

$= \frac{4}{5} \cdot \frac{1}{3} - \frac{-3}{5} \cdot \frac{\sqrt{8}}{3}$

$= \frac{4}{15} + \frac{3\sqrt{8}}{15} = \boxed{\frac{4+3\sqrt{8}}{15}}$

8. $\tan\beta$

$= \boxed{\sqrt{8}}$

9. $\sin(\alpha + \beta)$

$= \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta$

$= -\frac{3}{5} \cdot \frac{1}{3} + \frac{4}{5} \cdot \frac{\sqrt{8}}{3}$

$= \boxed{\frac{-3+4\sqrt{8}}{15}}$

10. $\sec\alpha$

$= \frac{1}{\cos\alpha}$

$= \frac{1}{\frac{4}{5}}$

$= \boxed{\frac{5}{4}}$

11. In which quadrant is α + β?

First! Both $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$ are positive.

12. Given $f(x) = x^2 - 3x + 5$, find the expression for *f'(x)* by definition.

$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - 3(x+\Delta x) + 5 - (x^2 - 3x + 5)}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{\cancel{x^2} + 2x\Delta x + \Delta x^2 \cancel{-3x} - 3\Delta x \cancel{+5} \cancel{-x^2} \cancel{+3x} \cancel{-5}}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{2x\cancel{\Delta x} + \Delta x^2 - 3\cancel{\Delta x}}{\cancel{\Delta x}} = \lim_{\Delta x \to 0} 2x + \Delta x - 3 = \boxed{2x-3}$

13. Find the average rate of change in the function $g(x) = 2x^2 - 3x + 1$ over the interval $-1 \le x \le 4$.

$g(-1) = 2(-1)^2 - 3(-1) + 1 = 6 \quad (-1, 6)$

$g(4) = 2(4)^2 - 3(4) + 1 = 21 \quad (4, 21)$

$AROC = \frac{21-6}{4--1} = \frac{15}{5} = \boxed{3}$

14. Find the equation of the line tangent to the curve $y = x^3 + 2x^2 - 5x - 1$ at $x = 2$.

$(2)^3 + 2(2)^2 - 5(2) - 1$

$= 8 + 8 - 11$

$= 5 \quad (2, 5)$

$\frac{dy}{dx} = 3x^2 + 4x - 5$

Slope $= 3(2)^2 + 4(2) - 5$

$= 12 + 8 - 5 = 15$

$\boxed{y - 5 = 15(x-2)}$

or

$\boxed{y = 15x - 25}$

15. Find the derivative, $\frac{dy}{dx}$, for each:

a. $y = 2x^{-3} - 9x + 5$

$\frac{dy}{dx} = -6x^{-4} - 9$

$= \boxed{\frac{-6}{x^4} - 9}$

b. $y = (2x-5)(x^2+5x+3)$

$\frac{dy}{dx} = (2x-5)(2x+5) + (x^2+5x+3)(2)$

$= 4x^2 + \cancel{10x} - \cancel{10x} - 25 + 2x^2 + 10x + 6$

$= \boxed{6x^2 + 10x - 19}$

c. $y = \frac{3x^2}{x^2-5x+3}$

$\frac{dy}{dx} = \frac{(x^2-5x+3)(6x) - (3x^2)(2x-5)}{(x^2-5x+3)^2}$

$= \frac{\cancel{6x^3} - 30x^2 + 18x - \cancel{6x^3} + 15x^2}{(x^2-5x+3)^2}$

$= \boxed{\frac{-15x^2 + 18x}{(x^2-5x+3)^2}}$

d. $y = \sqrt{4x-7}$ $\quad y = (4x-7)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}(4x-7)^{-\frac{1}{2}}(4)$

$= \frac{2}{(4x-7)^{\frac{1}{2}}}$

$= \boxed{\frac{2}{\sqrt{4x-7}}}$

e. $y = (4x^2 - 9x)^6$

$\frac{dy}{dx} = 6(4x^2-9x)^5(8x-9)$

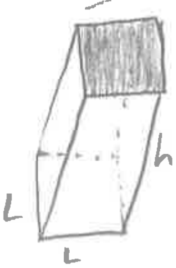
$= \boxed{(48x-54)(4x^2-9x)^5}$

f. $y = 4^{\pi}$

$\boxed{\frac{dy}{dx} = 0}$

Max

16. An open rectangular box with square base is to be made from 48 ft^2 of material. What dimensions will result in a box with the largest possible volume?



$V = L \cdot L \cdot h$

$V = L^2\left(\frac{48-L^2}{4L}\right)$

$V = \frac{48L - L^3}{4}$

$V = 12L - \frac{1}{4}L^3$

$V' = 12 - \frac{3}{4}L^2$

$48 = L^2 + 4Lh$

$48 - L^2 = 4Lh$

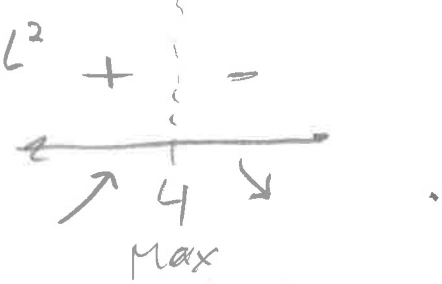
$\frac{48-L^2}{4L} = h$

$0 = 12 - \frac{3}{4}L^2$

$12 = \frac{3}{4}L^2$

$16 = L^2$

$L = \pm 4$



$h = \frac{48-4^2}{4 \cdot 4} = \frac{32}{16} = 2$

$\boxed{\begin{array}{l} h = 2 \text{ ft} \\ L = 4 \text{ ft.} \end{array}}$

Max 2 17. Consider a rectangle of perimeter 12 inches. Form a cylinder by revolving this rectangle about one of its edges. What dimensions of the rectangle will result in a cylinder of maximum volume?

$V = \pi r^2 h$ $\quad$ $12 = 2r + 2h$

$V = \pi r^2(6-r)$ $\quad$ $\frac{12-2r}{2} = h$

$V = \pi 6r^2 - \pi r^3$ $\quad$ $6 - r = h$

$V' = 12\pi r - 3\pi r^2$

$V' = 3\pi r(4-r)$

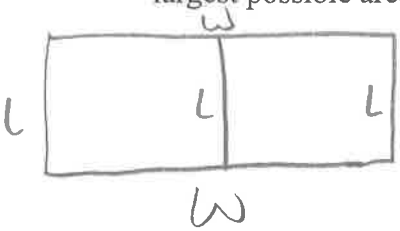
$r = 0, \; r = 4$

$3\pi r$	$-$	$+$	$+$
$4-r$	$+$	$+$	$-$
	$-$	0 $\quad +$	4 $\quad -$

Max

$h = 6 - 4$ $\quad$ $\boxed{h = 2 \text{ in}, \; r = 4 \text{ in}}$

18. A rancher wants to construct two identical rectangular corrals using 200 ft of fencing. The rancher decides to build them adjacent to each other, so they share fencing on one side. What dimensions should the rancher use to construct each corral so that together, they will enclose the largest possible area?



$A = Lw$ $\quad$ $200 = 3L + 2w$

$A = L(100 - 1.5L)$ $\quad$ $200 - 3L = 2w$

$A = 100L - 1.5L^2$ $\quad$ $100 - 1.5L = w$

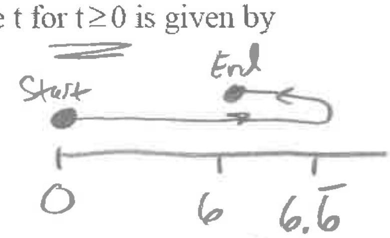
$A' = 100 - 3L$

$0 = 100 - 3L$

$\boxed{L = 33.\overline{3} \text{ ft}, \; w = 50 \text{ ft}}$ $\quad$ $w = 100 - 1.5(33.\overline{3}) = 50$

19. A particle moves along the x-axis in such a way that its position at time t for $t \geq 0$ is given by $p(t) = \frac{1}{3}t^3 - 3t^2 + 8t$.

a) Show that at time t = 0 the particle is moving to the right.
b) Find all values of t for which the particle is moving to the left.
c) What is the position of the particle at time t = 3?
d) When t = 3, what is the total distance the particle has traveled?



a) $v(t) = t^2 - 6t + 8$

$v(0) = 0^2 - 6(0) + 8$

$= \boxed{8}$ Moving right since positive

b) $0 = (t-4)(t-2)$

$t-4$	$-$	$-$	$+$
$t-2$	$-$	$+$	$+$
	$+$	2 $\quad -$	4 $\quad +$

$\boxed{\text{Left: } 2 < t < 4}$

c) $p(3) = 6$

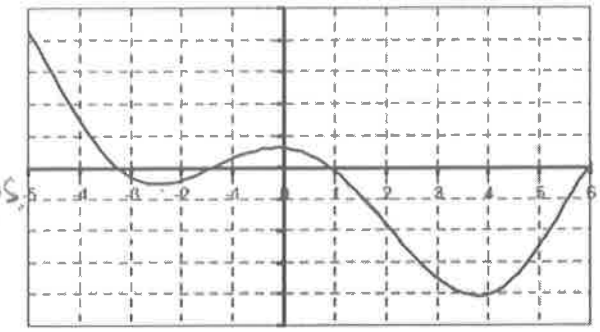
d) Start: $p(0) = 0$

Change direction: $p(2) = 6.\overline{6}$

End $p(3) = 6$

Distance: $6.\overline{6} + 0.\overline{6} = \boxed{7\tfrac{1}{3} \text{ units}}$

$f'(x)$



20. A graph of $f'(x)$ is given at the right.

A. On what interval(s) is $f(x)$ increasing? Decreasing? Explain.

Incr: $x < -3.3$, $-1.5 < x < 1$ $f'(x)$ is pos.

Decr: $-3.3 < x < -1.5$, $1 < x < 6$ $f'(x)$ is neg.

B. On what interval(s) is $f'(x)$ increasing? Decreasing? Explain. ↳ umm, duh

Incr: $-2.5 < x < 0$, $x > 3.8$

Decr: $x < -2.5$, $0 < x < 3.8$

C. On what interval(s) is $f(x)$ concave up? Concave down? Explain.

C.U.: $-2.5 < x < 0$, $x > 3.8$ $f'(x)$ is incr.

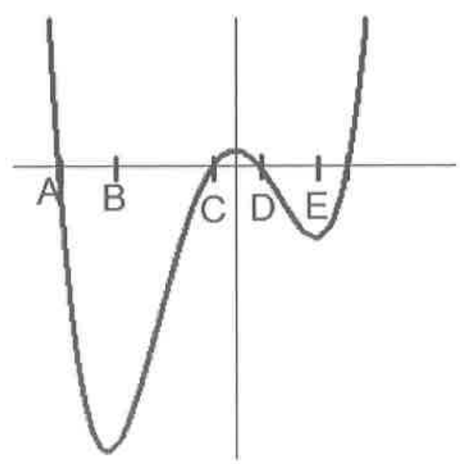
C.D.: $x < -2.5$, $0 < x < 3.8$ $f'(x)$ is decr.

D. On what interval(s) is $f'(x)$ concave up? Concave down? Explain. $\cup$ $\cap$

C.U.: $x < -1.5$, $x > 2$

C.D.: $-1.5 < x < 2$

21. At the right is a graph of $f(x)$. Give the interval(s) or point(s) where $f'(x)$ is negative, positive and zero.



Negative: $x < B$, $0 < x < E$

Positive: $B < x < 0$, $x > E$

Zero: $x = B$, $x = 0$, $x = E$

22. A particle moves on the x-axis (in units) such that its position at time t (in seconds) is given by the function:

$$s(t) = t^3 - 9t^2 + 15t, \quad 0 \le t \le 6$$

a. Determine the velocity & acceleration of the particle at time t.

$V(t) = 3t^2 - 18t + 15$

$a(t) = 6t - 18$

b. For what values of t is the particle at rest?

↳ $V(t) = 0$

$0 = 3t^2 - 18t + 15$

$0 = 3(t^2 - 6t + 5)$

$0 = 3(t-5)(t-1)$

$\boxed{t = 1 \text{ second } \& \ 5 \text{ seconds}}$

c. For what values of t is the particle moving to the right? To the left?

$t-1$	$-$	$+$	$+$
$t-5$	$-$	$-$	$+$
	0 (+) 1	(−) 5	(+) 6

Right: $0 \le t < 1$, $5 < t \le 6$

Left: $1 < t < 5$

d. What is the total distance it has traveled after 6 seconds?

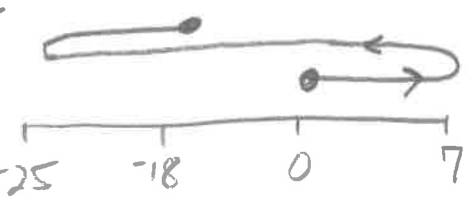
Start: $S(0) = 0$

Change direction: $S(1) = 7$, $S(5) = -25$

End: $S(6) = -18$

Distance $= 7 + 32 + 7$

$= \boxed{46 \text{ units}}$



e. What is the velocity when the acceleration is zero? Explain what your answer means in context.

$0 = 6t - 18$

$t = 3$

$V(3) = 3(3)^2 - 18(3) + 15$

$= \boxed{-12 \text{ units/s}}$

Moving left at a speed of 12 units/s.

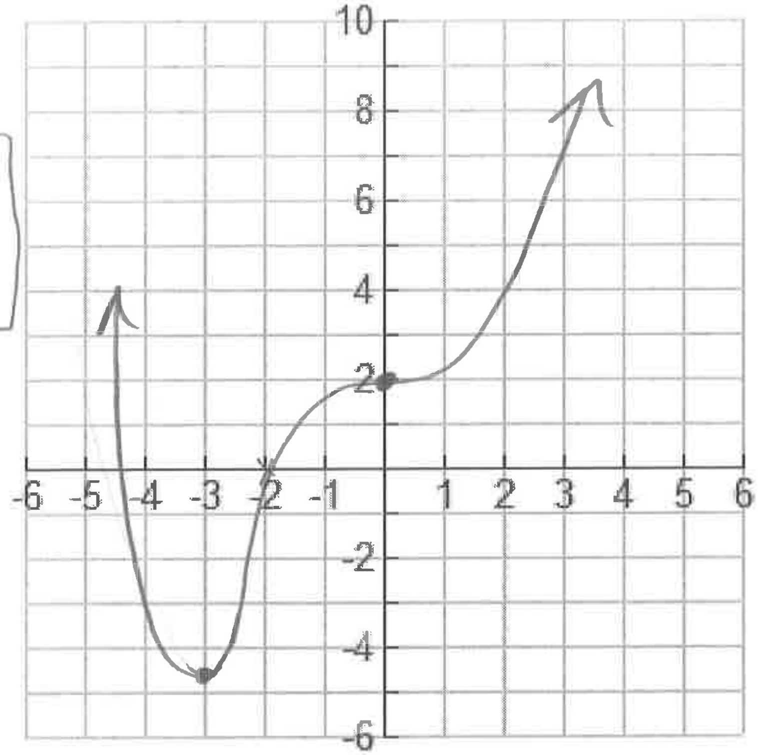
f. Is the particle speeding up or slowing down when $t = 4$ seconds?

$V(4) = -9$

$a(4) = 6$

Slowing down since the velocity & acceleration have opposite signs.

23. Consider the equation $f(x) = \frac{1}{4}x^4 + x^3 + 2$



a. Determine the points where there are maximums, minimums, or "flat spots".

$f'(x) = x^3 + 3x^2$

$0 = x^2(x+3)$

$x = 0, \; x = -3$

x^2	+	+	+
$x+3$	−	+	+
	−	+	+

$-3 \qquad 0$

Min at $x = -3$

flat spot at $x = 0$

b. Find the coordinates of the maximums, minimums, and "flat spots"

Min: $f(-3) = -\frac{19}{4}$ $\quad \boxed{\left(-3, -\frac{19}{4}\right)}$

Flat spot: $f(0) = 2$ $\quad \boxed{(0, 2)}$

c. Determine the concavity of $f(x)$. Your answer should be intervals.

$f''(x) = 3x^2 + 6x$

$0 = 3x(x+2)$

$x = 0 \qquad x = -2$

$3x$	−	−	+
$x+2$	−	+	+
	+	−	+

$-2 \qquad 0$

Concave up: $x < -2, \; x > 0$

Concave down: $-2 < x < 0$

d. Find the coordinates of the inflection point(s) of $f(x)$.

↳ $f''(x) = 0$

$(0, 2)$ + $(-2, -2)$

$f(0) \qquad f(-2)$

e. Sketch the graph of $f(x)$ based on the above information. Label all points you found above. DO NOT USE YOUR CALCULATOR!!

Math 4 – Final Exam Formulas

Pythagorean Identities

$$\cos^2 x + \sin^2 x = 1$$

$$\tan^2 x + 1 = \sec^2 x$$

$$\cot^2 x + 1 = \csc^2 x$$

Sum and Difference Identities

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

Double Angle Identities

$$\sin(2x) = 2\sin x \cos x$$

$$\cos(2x) = \cos^2 x - \sin^2 x$$

$$\cos(2x) = 2\cos^2 x - 1$$

$$\cos(2x) = 1 - 2\sin^2 x$$

$$\tan(2x) = \frac{2\tan x}{1 - \tan^2 x}$$

$$\tan(2x) = \frac{\sin(2x)}{\cos(2x)}$$